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OSCULATION BY ALGEBRAIC HYPERSURFACES

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Introduction

In this paper we give necessary and sufficient conditions for d pieces of hypersurface to be osculated to a fixed order by an algebraic hypersurface of degree d.

Given a line L_0 in P^{n+1} and d points $P_1^0,\cdots,P_d^0$ on L_0, suppose there are d pieces of hypersurface $\gamma_1,\cdots,\gamma_d$ such that $P_i^0 \in \gamma_i$ and L_0 intersects each γ_i transversely.

The question addressed here is: when does there exist an algebraic hypersurface γ of degree d which osculates each piece γ_i to order r at P_i^0? The main result gives necessary and sufficient conditions for the existence of such an algebraic hypersurface γ.

Fix affine coordinates $(x_0,\cdots,x_n)$ on P^{n+1}, and fix line coordinates $(m_1,\cdots,m_n, b_1,\cdots,b_n)$, where a line L is given by $x_k = m_k x_0 + b_k$, $k = 1,2,\cdots,n$. (Line coordinates are just local coordinates on $\mathrm{Gr}(1, n+1)$, the Grassmannian of all lines in P^{n+1}.) Assume that coordinates have been chosen so that the given line L_0 has line coordinates $m_k = 0$, $b_k = 0$ for all k. L_0 is then the x_0-axis. For convenience, write $m = (m_1,\cdots,m_n)$, $b = (b_1,\cdots,b_n)$. A line $L = L(m, b)$ near L_0 will intersect each γ_i at a point $P_i = P_i(m, b)$. Then $P_i(0,0) = P_i^0$. Let $X_i = X_i(m, b)$ be the 0th coordinate of P_i in terms of the affine coordinate system. Define $K_{jk} = K_{jk}(m, b)$ by

$$K_{jk}(m, b) = \frac{\partial^2[\Sigma_i X_i(m, b)]}{\partial b_j \partial b_k}$$

$j, k = 1,2,\cdots,n$. We can now state the main result.

Theorem. *There exists an algebraic hypersurface γ of degree d, which osculates each γ_i to order r, $2 \leqslant r \leqslant d$, at P_i^0, $i = 1,2,\cdots,d$, if and only if K_{jk} and all of its partial derivatives of order $\leqslant r-2$ vanish at $(m, b) = (0,0)$.*

Remarks. 1. If the order of osculation desired is $r = 0$ or $r = 1$, there is no condition. Just take γ to be the union of the d tangent hyperplanes to γ_i at P_i^0.

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2. In order to handle the case $r \geqslant d$, one must make assumptions on the derivatives of the individual X_i's not just on their sum. The needed conditions are obvious, and we will not concern ourselves with them.

3. The case of second-order osculations for curves in the plane ($n = 1$, $r = 2$) is treated by Griffiths-Harris [1, pp. 698–699]. The single condition obtained in that case is called the *Reiss relation*. The conditions of the theorem above may be considered as generalized Reiss relations.

Proof. The proof is in two parts. Each part will be treated in a separate section below. The necessity of the conditions follows from a direct computation along the lines of some earlier work of the author [3]. The sufficiency of the conditions will follow from a cohomological argument which counts the number of obstructions to the existence of γ. This cohomology argument is a direct generalization of the treatment by Griffiths-Harris of the $n = 1$, $r = 2$ case, as mentioned above.

Necessity. Suppose there exists an algebraic hypersurface γ of degree d, which osculates each piece of hypersurface γ_i to order r at P_i^0. In addition, suppose that γ has defining equation

$$p(x_0,\cdots,x_n) = 0,$$

where p is a polynomial degree d, and that γ_i has local defining equation

$$\phi_i(x_0,\cdots,x_n) = 0.$$

Because L_0 ($= x_0$-axis) meets γ and each γ_i transversely, the above equations define x_0 implicitly as a function of $x_1,\cdots,x_n$ locally near each P_i^0; say, for γ near P_i^0,

$$x_0 = \pi_i(x_1,\cdots,x_n) \qquad (\text{defining } \pi_i),$$

and for γ_i near P_i^0,

$$x_0 = \Phi_i(x_1,\cdots,x_n) \quad (\text{defining } \Phi_i).$$

Remember that our choice of coordinates dictates that $\pi_i(0,\cdots,0) = \Phi_i(0,\cdots,0) = X_i(0,0)$. Osculation to order r now means that the function $\pi_i - \Phi_i$ vanishes through order r at $(x_1,\cdots,x_n) = (0,\cdots,0)$. Recall that $X_i(m,b)$ is the 0th coordinate of the point of intersection of γ_i and $L(m,b)$. Now define $Y_i = Y_i(m,b)$ to be the 0th coordinate of the point of intersection of γ and $L(m,b)$ near the point of intersection of γ_i and $L(m,b)$. Because γ osculates γ_i, Y_i will be close to X_i for (m,b) near $(0,0)$. More precisely, we have the following lemma.

Lemma. *If γ osculates γ_i to order r at P_i^0, then the function $X_i(m,b) - Y_i(m,b)$ vanishes up through order r at $(m,b) = (0,0)$.*

Proof. $X_i - Y_i$ vanishes (to order zero) at $(0,0)$ because $X_i(0,0) = \Phi_i(0,\cdots,0)$, $Y_i(0,0) = \pi_i(0,\cdots,0)$, and $\Phi_i(0,\cdots,0) = \pi_i(0,\cdots,0)$.

Before proceeding to the higher orders, observe that $X_i(m, b) = \Phi_i(mX_i(m, b) + b)$, where $mX_i(m, b) + b$ is short-hand for $(m_1 X_i(m, b) + b_1, \cdots, m_n X_i(m, b) + b_n)$. Similarly, $Y_i(m, b) = \pi_i(mY_i(m, b) + b)$.

Let ∂_m^J, ∂_b^K be multi-indexed partial derivatives with respect to the m- and b-variables respectively.

As an induction hypothesis, suppose $X_i - Y_i$ vanishes up through order s $(< r)$ at $(0,0)$. Consider one of the $(s+1)$st order partial derivatives $\partial[\partial_m^J \partial_b^K (X_i - Y_i)]/\partial m_l$, where $|J| + |K| = s$.

$$\begin{aligned}\frac{\partial}{\partial m_l}&[\partial_m^J \partial_b^K (X_i - Y_i)]\\ &= \partial_m^J \partial_b^K \left[\frac{\partial}{\partial m_l} (\Phi_i(mX_i(m, b) + b)) - \frac{\partial}{\partial m_l} (\pi_i(mY_i(m, b) + b)) \right]\\ &= \partial_m^J \partial_b^K \left[\sum_j \frac{\partial \Phi_i}{\partial x_j} \left(\delta_{jl} X_i + m_j \frac{\partial X_i}{\partial m_l} \right) - \sum_j \frac{\partial \pi_i}{\partial x_j} \left(\delta_{jl} + m_j \frac{\partial Y_i}{\partial m_l} \right) \right],\end{aligned}$$

where $\partial \Phi_i / \partial x_j$ and $\partial \pi_i / \partial x_j$ are evaluated at $mX_i + b$ and $mY_i + b$ respectively.

The remaining derivatives remain to be calculated, but are easily visualized. Because of the product rule, any $(s+1)$st order derivative of either X_i or Y_i is multiplied by an m_j. After evaluating the expression at $(m, b) = (0,0)$, such terms vanish. The remaining terms are expressible in terms of derivatives of Φ_i and π_i to orders $\leqslant s + 1 \leqslant r$ and derivatives of X_i and Y_i to orders $\leqslant s$. Moreover, the entire expression is clearly skew-symmetric in the pairs (Φ_i, X_i) and (π_i, Y_i). When we evaluate at $(m, b) = (0,0)$, the Φ_i, π_i terms are evaluated at mX_i [or Y_i] $+ b = 0$. $\Phi_i - \pi_i$ vanishes to order r at $(0, \cdots, 0)$ by the osculation assumption, and $X_i - Y_i$ vanishes through order s at $(0,0)$ by induction. Thus the expression is also symmetric in the pairs (Φ_i, X_i) and (π_i, Y_i), and consequently vanishes.

A similar argument holds for $\partial[\partial_m^J \partial_b^K (X_i - Y_i)]/\partial b_l$.

The necessity of the conditions in the main theorem then follows from the following theorem which was first proved in [3]. Since its proof is very simple, we include it here for convenience.

Theorem. *If γ is an algebraic hypersurface in P^{n+1} of degree d, and if $Y_i(m, b)$ $(i = 1, \cdots, d)$ are the 0th coordinates of the intersection points of γ with a line $L(m, b)$, then*

$$\frac{\partial^2}{\partial b_j \partial b_k} \sum_i Y_i(m, b) = 0,$$

for all (m, b), and for all $j, k = 1, \cdots, n$.

Proof. If γ satisfies the polynomial equation of degree d:

$$p(x_0,\cdots,x_n)=0,$$

then the Y_i's are the d solutions of the following equation (in x_0):

$$\tilde{p}(x_0)=p(x_0,m_1x_0+b_1,\cdots,m_nx_0+b_n)=0.$$

$\tilde{p}$ is a polynomial of degree d in x_0, whose coefficients are in turn polynomials in (m, b). If we set

$$\tilde{p}(x_0)=A_0(m,b)x_0^d-A_1(m,b)x_0^{d-1}+\cdots\pm A_d(m,b),$$

then the definition of $\tilde{p}$ in terms of p implies that A_0 is independent of b and that A_1 is linear in b.

It is well known that the sum of the roots of $\tilde{p}$, namely $\Sigma Y_i(m,b)$, is expressible in terms of the coefficients of $\tilde{p}$. Specifically,

$$\sum Y_i(m,b)=A_1/A_0.$$

Since A_1/A_0 is only linear in b, the second b-partials must vanish.

Sufficiency. To prove sufficiency, we generalize the approach in Griffiths-Harris [1, pp. 698–699]. The author would like to thank Phillip Griffiths, Lawrence Ein and Raghavan Narasimhan for their helpful comments and advice.

Henceforth we shall denote P^{n+1} simply by P. Let I be the ideal sheaf of the line L_0 in P. Let O be the structure sheaf of P, and let $O_s=O/I^{s+1}$. In particular, O_0 is the structure sheaf of L_0. Define the *sth infinitesimal neighborhood of* L_0 to be the scheme $L_s=(L_0,O_s)$. Let Pic(V) denote the set of Cartier divisors on a scheme V, modulo linear equivalence. Pic(V) is isomorphic to $H^1(V,O_V^*)$.

An sth order germ of hypersurface intersecting L_0 defines an element η of Pic(L_s), together with a section of η. There is also a natural restriction map $a_s:\text{Pic}(P)\to\text{Pic}(L_s)$ obtained by intersection with L_s. Our problem of determining when there exists an algebraic hypersurface γ which osculates γ_i to rth order at P_i^0 is equivalent to first determining when an element $D\in\text{Pic}(L_r)$ is in the image of the map $a_r:\text{Pic}(P)\to\text{Pic}(L_r)$, and then determining whether sections of D lift.

Using $\text{Pic}(P)\cong H^1(P,O^*)$ and $\text{Pic}(L_r)\cong H^1(L_r,O_r^*)$, information about the image of $a_r\colon H^1(P,O^*)\to H^1(L_r,O_r^*)$ can be obtained in the following manner. From the exact sequence

$$0\to I^{r+1}\to O\to O_r\to 0,$$

exponentiate to obtain the multiplicativve version

$$0\to J_{r+1}\to O^*\to O_r^*\to 0.$$

The long exact sequence on cohomology includes

$$(*) \qquad \cdots \to H^1(P,O^*) \to H^1(L_r,O_r^*) \to H^2(P,J_{r+1}) \to H^2(P,O^*) \to \cdots.$$

It is easy to see that $H^2(P,O^*)=0$. Indeed, from the exact exponential sheaf sequence

$$0 \to Z \to O \to O^* \to 0,$$

follows the long exact cohomology sequence

$$\cdots \to H^2(P,O) \to H^2(P,O^*) \to H^3(P,Z) \to \cdots.$$

But $H^2(P,O)=0$ and $H^3(P,Z)=0$, so that $H^2(P,O^*)=0$ also. Thus sequence $(*)$ above becomes

$$\cdots \to H^1(P,O^*) \to H^1(L_r,O_r^*) \to H^2(P,J_{r+1}) \to 0,$$

where the first map is a_r. Thus the obstructions to $D \in H^1(L_r,O_r^*)$ being in the image of a_r lie in $H^2(P,J_{r+1})$. Since $J_{r+1} \cong I^{r+1}$, we will next calculate $h^2(P,I^{r+1})$.

Let $N^* = I/I^2$ be the conormal bundle of L_0 in P. Because L_0 is just a line in P, $N^* \cong O_0^n(-1)$ where the superscript n means to take the direct sum of n copies of $O_0(-1)$.

Proposition. *$H^i(P^{n+1},I) = H^i(P^{n+1},I^2) = 0$ for $i \geqslant 0$. For $r \geqslant 2$, $H^i(P^{n+1},I^{r+1}) = 0$, $i \neq 2$, and*

$$h^2(P^{n+1},I^{r+1}) = \binom{n+1}{2} + 2\binom{n+2}{3} + \cdots + (r-1)\binom{n-1+r}{r}.$$

Proof. First, from the definition of I we have the exact sheaf sequence $0 \to I \to O \to O_0 \to 0$, where, as above, O is the structure sheaf of $P = P^{n+1}$, and O_0 is the structure sheaf of the line L_0. On cohomology we have

$$\begin{aligned}0 \to H^0(P,I) \to H^0(P,O) \to H^0(L_0,O_0)\\ \to H^1(P,I) \to H^1(P,O) \to H^1(L_0,O_0) \to \cdots.\end{aligned}$$

But $H^0(P,O) \cong H^0(L_0,O_0) \cong \{\text{constants}\}$, and $H^i(P,O) \cong H^i(L_0,O_0) = 0$ for $i \geqslant 1$, so that $H^i(P,I) = 0$ for all $i \geqslant 0$.

Second, note the exact sequence $0 \to I^2 \to I \to N^* \to 0$, which induces the long exact sequence

$$\cdots \to H^{i-1}(L_0,N^*) \to H^i(P,I^2) \to H^i(P,I) \to H^i(L_0,N^*) \to \cdots.$$

Because $H^i(L_0,O_0(-1)) = 0$ for all $i \geqslant 0$, we see that $H^i(P,I) \cong H^i(P,I^2)$. So $H^i(P,I^2) = 0$ for all $i \geqslant 0$.

We prove the remainder of the proposition by induction on r. For the case $r=2$ we use that

$$I^2/I^3 \cong \mathrm{Sym}^2(N^*) \cong O_0^{\mathrm{rk}(2)}(-2),$$

where $\mathrm{rk}(2) = \mathrm{rank}(\mathrm{Sym}^2(N^*))$, namely, $\binom{n+1}{2}$. Then

$$0 \to I^3 \to I^2 \to \mathrm{Sym}^2(N^*) \to 0$$

induces

$$\cdots \to H^1(P, I^2) \to H^1(L_0, \mathrm{Sym}^2(N^*)) \to H^2(P, I^3)$$
$$\to H^2(P, I^2) \to H^2(L_0, \mathrm{Sym}^2(N^*)) \to \cdots.$$

Because $H^i(P, I^2) = 0$ for all $i \geqslant 0$, and $H^i(L_0, O_0(-2)) = 0$ for $i \neq 1$, we see that $H^2(P, I^3) \cong H^1(L_0, \mathrm{Sym}^2(N^*))$ and that $H^i(P, I^3) = 0$ for $i \neq 2$. By Serre duality, $h^1(L_0, O_0(-2)) = h^0(L_0, O_0) = 1$. Thus $h^2(P, I^3) = 1 \cdot \mathrm{rk}(2) = \binom{n+1}{2}$.

For the induction step, we assume the result for $r = s$ and prove it for $r = s + 1$. Using the isomorphism

$$I^s/I^{s+1} \cong \mathrm{Sym}^s(N^*) \cong O_0^{\mathrm{rk}(s)}(-s),$$

where $\mathrm{rk}(s) = \mathrm{rank}(\mathrm{Sym}^s(N^*)) = \binom{n-1-s}{s}$, we get the exact sheaf sequence

$$0 \to I^{s+1} \to I^s \to \mathrm{Sym}^s(N^*) \to 0.$$

The long exact cohomology sequence contains

$$\cdots \to H^1(P, I^s) \to H^1(L_0, \mathrm{Sym}^s(N^*)) \to H^2(P, I^{s+1})$$
$$\to H^2(P, I^s) \to H^2(L_0, \mathrm{Sym}^s(N^*)) \to \cdots.$$

Because $h^1(L_0, O_0(-s)) = h^0(L_0, O_0(s-2)) = s - 1$, and all other $h^i(L_0, O_0(-s)) = 0$ for $i \neq 1$, the induction hypothesis on $h^i(P, I^s)$ forces $h^i(P, I^{s+1}) = 0$ for $i \neq 2$, and the long exact sequence collapses to

$$0 \to H^1(L_0, \mathrm{Sym}^s(N^*)) \to H^2(P, I^{s+1}) \to H^2(P, I^s) \to 0.$$

Then

$$h^2(P, I^{s+1}) = h^2(P, I^s) + h^1(L_0, \mathrm{Sym}^s(N^*))$$
$$= h^2(P, I^s) + (s-1)\mathrm{rk}(s) = h^2(P, I^s) + (s-1)\binom{n-1+s}{s},$$

and the induction step is complete.

We conclude the first part of the proof of sufficiency by showing that the number of independent conditions among the necessary conditions (i.e., that K_{jk} and its partial derivatives through order $r-2$ vanish at $(m, b) = 0$) is exactly equal to the dimension of the space of obstructions $H^2(P, I^{r+1})$.

For $r = 2$, $h^2(P, I^3) = \binom{n+1}{2}$. On the other hand $K_{jk} = K_{kj}$, so that the number of independent components is also $\binom{n+1}{2}$.

By induction, it suffices to prove that the number of independent partial derivatives of K_{jk} of order exactly $s-2$ is equal to

$$h^2(P, I^{s+1}) - h^2(P, I^s) = h^1(L_0, \mathrm{Sym}^s(N^*)) = (s-1)\binom{n-1+s}{s}.$$

At first glance, this appears hopeless because even in the case $s = 3$, the first-order partial derivatives of K_{jk} are

$$\partial\left[K_{jk}\right]/\partial m_l = \partial^3\left[\sum X_i\right]/\partial b_j \partial b_k \partial m_l,$$
$$\partial\left[K_{jk}\right]/\partial b_l = \partial^3\left[\sum X_i\right]/\partial b_j \partial b_k \partial b_l,$$

of which there appear to be $\binom{n+2}{3} + n\binom{n+1}{2}$ which are independent.

Fortunately, there are relations among the partial derivatives of ΣX_i.

Recall that $X_i(m, b)$ is the 0th coordinate of the point of intersection of $L(m, b)$ with γ_i, that γ_i has defining equation $\phi_i(x_0, \cdots, x_n) = 0$ and that $L(m, b)$ satisfies $x_k = m_k x_0 + b_k$ for $k = 1, \cdots, n$. Thus X_i satisfies the identity

$$\phi_i(X_i, m_1 X_i + b_1, \cdots, m_n X_i + b_n) = 0,$$

for all (m, b). Differentiation of this identity with respect to m_j yields

$$(\partial\phi_i/\partial x_0)(\partial X_i/\partial m_j) + \sum_k \left[(\partial\phi_i/\partial x_k)(m_k \partial X_i/\partial m_j + \delta_{kj} X_i)\right] = 0,$$

while differentiation with respect to b_j yields

$$(\partial\phi_i/\partial x_0)(\partial X_i/\partial b_j) + \sum_k \left[(\partial\phi_i/\partial x_k)(m_k \partial X_i/\partial b_j + \delta_{kj})\right] = 0.$$

From these two equations it follows that

$$\partial X_i/\partial m_j = X_i(\partial X_i/\partial b_j).$$

Equipped with this relation, let us reexamine the case $s = 3$:

$$\begin{aligned}
\frac{\partial\left[K_{jk}\right]}{\partial m_l} &= \frac{\partial^3[\Sigma X_i]}{\partial b_j \partial b_k \partial m_l} = \frac{\partial^2[\Sigma X_i(\partial X_i/\partial b_l)]}{\partial b_j \partial b_k} \\
&= \frac{\partial}{\partial b_j}\left(\sum \frac{\partial X_i}{\partial b_k}\frac{\partial X_i}{\partial b_l} + \sum X_i \frac{\partial^2 X_i}{\partial b_k \partial b_l}\right) \\
&= \sum X_i \frac{\partial^3 X_i}{\partial b_j \partial b_k \partial b_l} + \sum\left(\frac{\partial X_i}{\partial b_j}\frac{\partial^2 X_i}{\partial b_k \partial b_l} + \frac{\partial X_i}{\partial b_k}\frac{\partial^2 X_i}{\partial b_j \partial b_l} + \frac{\partial X_i}{\partial b_l}\frac{\partial^2 X_i}{\partial b_j \partial b_k}\right),
\end{aligned}$$

$$\frac{\partial\left[K_{jk}\right]}{\partial b_l} = \frac{\Sigma\partial^3 X_i}{\partial b_j \partial b_k \partial b_l}.$$

Now it is clear that there are only $2\binom{n-1+3}{3} = h^1(L_0, \mathrm{Sym}^s(N^*))$ independent components among the first partial derivatives of K_{jk}. The form of the general derivative of K_{jk} of order s is now predictable.

Proposition. *Let E and F be multi-indices with $|E| = e$ and $|F| = f$. Let $G = E + F$. If $e + f = s$, then*

$$\partial^s[K_{jk}]/\partial m^E \partial b^F \equiv \sum X_i^e \left(\partial^{s+2} X_i / \partial b^G \partial b_j \partial b_k\right),$$

modulo terms containing derivatives of X_i of order $1, \cdots, s-1$.

Proof. For $s = 0, 1$, this has already been shown above. The induction step is quite easy. If $e + f = s$, then

$$\frac{\partial^{s+1}[K_{jk}]}{\partial m_l \partial m^E \partial b^F} \equiv \frac{\partial}{\partial m_l}\left[\sum X_i^e \frac{\partial^{s+2} X_i}{\partial b^G \partial b_j \partial b_k}\right]$$

$$\equiv \left\{\sum e X_i^{e-1} X_i \frac{\partial X_i}{\partial b_l} \frac{\partial^{s+2} X_i}{\partial b^G \partial b_j \partial b_k}\right\} + \sum X_i^e \frac{\partial^{s+2}[X_i \partial X_i / \partial b_l]}{\partial b^G \partial b_j \partial b_k}$$

$$\equiv \sum X_i^{e+1} \frac{\partial^{s+3} X_i}{\partial b^G \partial b_j \partial b_k \partial b_l},$$

modulo derivatives of order $\leqslant s$.

A similar computation holds for $\partial^{s+1}[K_{jk}]/\partial b_l \partial m^E \partial b^F$.

We next note that

$$\frac{\partial^2 X_i}{\partial m_j \partial b_k} = \frac{\partial}{\partial b_k}\left[\frac{X_i \partial X_i}{\partial b_j}\right] = X_i \frac{\partial^2 X_i}{\partial b_j \partial b_k} + \frac{\partial X_i}{\partial b_j}\frac{\partial X_i}{\partial b_k}$$

is symmetric in j and k. Thus the expression for $\partial^s[K_{jk}]/\partial m^E \partial b^F$, once expressed entirely in terms of b-derivatives of X_i, is a symmetric expression in the $s+2$ indices which appear. However, the highest order terms are distinguished by the power of X_i appearing as a factor. That power is precisely e. Because e can be any number from 0 to s, there will be $s+1$ independent groups of derivatives with each group in turn depending upon $s+2$ symmetric indices. Thus the total number of independent components among the derivatives of K_{jk} of order s is $(s+1)\binom{n-1+s+2}{s+2}$. For $s = r-2$, the number of independent components is $(r-1)\binom{n-1+r}{r}$ which is precisely $h^1(L_0, \mathrm{Sym}^s(N^*))$.

The number of independent necessary conditions being equal to the number of obstructions to $D \in \mathrm{Pic}(L_r)$ being in the image of $a_r: \mathrm{Pic}(P) \to \mathrm{Pic}(L_r)$, the necessary conditions are also sufficient. This concludes the first part of the proof of sufficiency.

The second part of the proof of sufficiency is concerned with lifting sections. Our original data of d pieces of hypersurface meeting L_0 determine both a line bundle $D \in \operatorname{Pic}(L_r)$ and a section σ of D. Assuming the conditions stated in the theorem we have just proved the existence of a line bundle E on P^{n+1} such that E restricts to D. To get an actual hypersurface γ which osculates the original γ_i's, we must lift the section σ of D to a section τ of E.

Because there were d pieces of hypersurface and E restricts to a divisor linearly equivalent to D, the degree of E is also d. Thus $E \cong O(d)$ on P^{n+1}. The restriction map $E \to D \to 0$ has a kernel consisting of elements of $E \cong O(d)$ which vanish on L_r, namely $I^{r+1}(d)$. To prove that a section σ of D can be lifted to a section τ of E, it suffices to prove that $H^1(P^{n+1}, I^{r+1}(d)) = 0$. This will follow from the following proposition.

Proposition. *If $s \geq 1$ and $d \geq s - 2$, then $H^i(P^{n+1}, I^s(d)) = 0$, for $i \geq 1$.*

Proof. If P^{n+1} has homogeneous coordinates $[x_0, \cdots, x_{n+1}]$ such that our original affine coordinates are given by taking $x_{n+1} = 1$, then L_0 is cut out by the equations $x_1 = x_2 = \cdots = x_n = 0$, and $I(1)$ has global sections $x_1, \cdots, x_n$. A basis for the global sections of $I(d)$ consists of degree d monomials in $x_0, \cdots, x_{n+1}$ having at least one factor from $x_1, \cdots, x_n$.

Case $s = 1$. From the exact sheaf sequence on P^{n+1},

$$0 \to I \to O \to O_0 \to 0,$$

we twist by d to get

$$0 \to I(d) \to O(d) \to O_0(d) \to 0.$$

Since L_0 is a line in P^{n+1}, $O_0 \otimes O(d) \cong O_0(d)$. For $d \geq -1$, $H^i(L_0, O_0(d)) = H^i(P^{n+1}, O(d)) = 0$ for $i \geq 1$. So the long exact sequence on cohomology implies that $H^i(P^{n+1}, I(d)) = 0$ for $i \geq 2$, and thus the exact cohomology sequence collapses to

$$0 \to H^0(P, I(d)) \to H^0(P, O(d)) \to H^0(L_0, O_0(d)) \to H^1(P, I(d)) \to 0,$$

where, as above, $P = P^{n+1}$.

If $d = -1$, all of the H^0-groups above vanish, so $H^1(P, I(-1)) = 0$. For $d = 0$, $H^1(P, I) = 0$ by an earlier proposition. For $d \geq 1$, a basis of $H^0(P, O(d))$ consists of all monomials in $x_0, \cdots, x_{n+1}$ of degree d. A basis for $H^0(P, I(d))$ consists of all those monomials of degree d containing at least one factor from $x_1, \cdots, x_n$. Thus a basis of $H^0(P, I(d))$ can be extended to a basis of $H^0(P, O(d))$ by including those monomials of degree d in x_0, x_{n+1} only, which is precisely a basis of $H^0(L_0, O_0(d))$. Thus $h^0(P, O(d)) = h^0(P, I(d)) + h^0(L_0, O_0(d))$, and we conclude that $H^1(P, I(d)) = 0$.

Case $s \geq 2$. The proof now proceeds by induction on s. We assume the result for $s \geq 1$ and prove it for $s + 1 \geq 2$. In particular, we assume $H^i(P, I^s(d)) = 0$ for $d \geq s - 2$, $i \geq 1$. Using the exact sheaf sequence

$$0 \to I^{s+1} \to I^s \to \operatorname{Sym}^s N^* \to 0,$$

we twist by $d \geqslant (s+1) - 2 = s - 1$ to obtain

$$(**) \qquad 0 \to I^{s+1}(d) \to I^s(d) \to (\mathrm{Sym}^s N^*)(d) \to 0.$$

Now $\mathrm{Sym}^s N^* \cong O_0^{\mathrm{rk}(s)}(-s)$, where $\mathrm{rk}(s) = \mathrm{rank}(\mathrm{Sym}^s N^*) = \binom{n-1+s}{s}$, so that $(\mathrm{Sym}^s N^*)(d) \cong O_0^{\mathrm{rk}(s)}(d-s)$. Because we assume $d \geqslant (s+1) - 2 = s - 1$, $d - s \geqslant -1$, so that $H^i(L_0, O_0(d-s)) = 0$ for $i \geqslant 1$. Then the long exact cohomology sequence obtained from $(**)$ becomes

$$0 \to H^0(P, I^{s+1}(d)) \to H^0(P, I^s(d)) \to H^0(L_0, (\mathrm{Sym}^s N^*)(d))$$
$$\to H^1(P, I^{s+1}(d)) \to H^1(P, I^s(d)) \to 0,$$

together with

$$0 \to H^i(P, I^{s+1}(d)) \to H^i(P, I^s(d)) \to 0,$$

for $i \geqslant 2$. By induction, $H^i(P, I^s(d)) = 0$, $d \geqslant s - 2$, $i \geqslant 1$, so that $H^i(P, I^{s+1}(d)) = 0, i \geqslant 2, d \geqslant s - 1$.

We now wish to count the dimension of $H^0(P, I^r(d))$. A basis of $H^0(P, I^r(d))$ consists of monomials of degree d in $x_0, \cdots, x_{n+1}$ containing a factor of a monomial of degree r in $x_1, \cdots, x_n$. To complete this to a basis of all of $H^0(P, O(d))$, we must include monomials of degree d with fewer than r factors from $x_1, \cdots, x_n$. The number of such monomials is equal to

$$\sum \binom{n-1+j}{j}(1+d-j),$$

where the summation is over $j = 0, \cdots, r-1$. $\binom{n-1+j}{j}$ is the number of monomials in $x_1, \cdots, x_n$ of degree j, while $1 + d - j$ is the number of monomials in x_0, x_{n+1} of degree $d - j$. Thus $h^0(P, I^r(d)) = h^0(P, O(d)) - \Sigma(1 + d - j)\binom{n-1+j}{j}$. We apply this formula:

$$h^0(P, I^s(d)) - h^0(P, I^{s+1}(d)) = (1 + d - s)\binom{n-1+s}{s}.$$

But this is precisely equal to $h^0(L_0, (\mathrm{Sym}^s N^*)(d)) = \mathrm{rk}(s) \cdot h^0(L_0, O_0(d-s))$. So the long exact cohomology sequence breaks up once more, and $H^1(P, I^{s+1}(d)) \cong H^1(P, I^s(d))$, which vanishes by induction.

Using this proposition, we see that $H^1(P^{n+1}, I^{r+1}(d)) = 0$, provided $d \geqslant r - 1$. Since we have assumed $d \geqslant r$, there is no obstruction to lifting sections, and the proof of the sufficiency of the stated conditions is now complete.

Formula for γ. Assuming the conditions of the main theorem, we know of the existence of an algebraic hypersurface γ which osculates our original γ_i's. A curious fact is that one can actually write down the degree d polynomial defining γ. The proof that the γ so defined actually osculates the original γ_i's is not a pleasant task and will not be taken up here. For details, see [2].

In homogeneous coordinates $[x_0, \cdots, x_{n+1}]$, chosen so that the affine coordinates used above are given by $x_{n+1} = 1$, the polynomial of degree d defining γ is of the form

$$p(x_0, \cdots, x_{n+1}) = \sum_E \binom{d}{E} B_E x^E,$$

where the summation is over all multi-indices $E = (E(0), E(1), \cdots, E(n+1))$ of degree d, i.e., $\sum_i E(i) = d$, and where $\binom{d}{E} = d!/(E(0)! \cdots E(n+1)!)$ and $x^E = x_0^{E(0)} \cdots x_{n+1}^{E(n+1)}$.

Let S_k equal the kth elementary symmetric polynomial in $\{X_1, \cdots, X_d\}$. Set $S_0 = 1$. Then

$$B_E = (-1)^{d-E(0)} \frac{E(n+1)!}{\binom{d}{E(0)}(d-E(0))!} \frac{\partial^{d-E(n+1)}[S_{d-E(0)}]}{\partial b_1^{E(1)} \cdots \partial b_n^{E(n)}},$$

evaluated at $b = 0$, gives the coefficients of p.

References

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